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Universal Covering Group of $U(n)$ and Projective Representations

M. A. Aguilar[1] and M. Socolovsky[2]

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Using fiber bundle theory, we construct the universal covering group of $U(n)$, $\tilde{U}(n)$, and show that $\tilde{U}(n)$ is isomorphic to the semidirect product $SU(n) \circledS \mathbb{R}$. We give a bijection between the set of projective representations of $U(n)$ and the set of equivalence classes of certain unitary representations of $SU(n) \circledS \mathbb{R}$. Applying Bargmann's theorem, we give explicit expressions for the liftings of projective representations of $U(n)$ to unitary representations of $SU(n) \circledS \mathbb{R}$. For completeness, we discuss the topological and group theoretic relations between $U(n)$, $SU(n)$, $U(1)$, and $\mathbb{Z}_n$.

1. INTRODUCTION

The universal covering group $\tilde{G}$ of a connected Lie group G (Warner, 1983) and its associated projection $\tilde{G} \to G$ play an important role in many physical applications. This is due to the fact that $\tilde{G}$ has representations which do not come from representations of G, the so-called spinor representations of G. The most important and best known examples are the rotations in 3-dimensional Euclidean space, $SU(2) \to SO(3)$, and the Lorentz transformations in 4-dimensional Minkowski spacetime, $SL_2(\mathbb{C}) \to SO^0(3, 1)$.

In the Abelian case, the universal covering group of the circle, $\mathbb{R} \to U(1)$, is the vehicle through which virtual classical paths contribute to the Feynman path integral expression for the quantum transition amplitudes in nonrelativistic quantum mechanics (Feynman and Hibbs, 1965).

On the other hand, projective representations of symmetry groups appear naturally in quantum mechanics, since, on one hand, the pure states of any

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, 04510 México, D.F., Mexico; e-mail: marcelo@math.unam.mx.
[2] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, 04510 México, D.F., Mexico; e-mail: socolovs@nucleu.unam.mx.

physical system are represented by rays in the corresponding Hilbert space, and, on the other, the unitary or antiunitary operators representing the symmetry transformations are required to preserve transition probabilities, that is, the square of the modulus of the transition amplitudes, but not the transition amplitudes themselves; therefore the operators are determined only up to a phase. For Lie groups, however, these extra phases can be eliminated, that is, one can pass to a truly unitary representation, when the symmetry group satisfies the conditions of Bargmann's theorem (Bargmann, 1954), namely, the group should be simply connected and the second cohomology of its Lie algebra should be equal to zero (see Section 4). If the group is connected, but not simply connected, one passes to its universal covering group.

One should emphasize, however, that the quantum physical symmetry group is determined by the projective representation (Belinfante and Kolman, 1972).

In this paper we study the universal covering group $\tilde{U}(n)$ of $U(n)$, the automorphism group of the Hilbert space $\mathbb{C}^n$, and show that $\tilde{U}(n)$ is isomorphic to the semidirect product $SU(n) \circledS \mathbb{R}$ (Section 3). From the physical point of view, $U(n)$ is the internal symmetry group of a system of n identical noninteracting harmonic oscillators (Bargmann and Moshinsky, 1960).

In Section 4 we discuss some preliminary results on unitary and projective representations. In particular, we study the concept of strong continuity of a representation, and show that the group of unitary transformations of a Hilbert space $\mathcal{H}$, $\mathcal{U}(\mathcal{H})$, is a topological group.

In Section 5 we give a bijection between the set of projective representations of $U(n)$ and the set of orbits of certain unitary representations of $SU(n) \circledS \mathbb{R}$ under the action of the group of one-dimensional representations of $SU(n) \circledS \mathbb{R}$.

In Section 6 we construct all the unitary representations of $\tilde{U}(n)$ associated to a projective representation of $U(n)$.

As an introduction, we discuss, in Section 2, the topological and group theoretic relations between $U(n)$, $U(1)$, $SU(n)$, and $\mathbb{Z}_n$.

2. THE GROUPS $U(N)$

The unitary group $U(n)$, $n = 1, 2, \ldots,$ is the group of automorphisms of the Hilbert space $(\mathbb{C}^n, \langle\ ,\ \rangle)$ where $\langle\ ,\ \rangle$ is the Hermitian scalar product $\langle \vec{z}, \vec{w} \rangle = \sum_{k=1}^{n} z_k \overline{w}_k$. If $A \in U(n)$ and A^* is the transpose conjugate matrix, then $A^*A = I$, i.e., $A^* = A^{-1}$, so $|\det A| = 1$ and $\dim_{\mathbb{R}} U(n) = n^2$. Here $U(n)$ is a Lie group and $SU(n)$ is the closed Lie subgroup consisting of matrices whose determinant is 1. In particular $U(1)$ is the unit circle or 1-sphere S^1.

Let φ_n: $U(1) \times SU(n) \to U(n)$ be given by $\varphi_n(z, A) := zA$. Since $(zA)^* = \bar{z}A^* = (zA)^{-1}$, φ_n is well defined. Now we show that φ_n is a surjective Lie group homomorphism with kernel $\mathbb{Z}_n$: (i) $\varphi_n((z, A)(z', A')) = \varphi_n(zz', AA') = (zz')(AA') = (zA)(z'A') = \varphi_n(z, A)\varphi_n(z', A')$, i.e., φ_n is a group homomorphism; (ii) if $B \in U(\mathrm{n})$, then $(detB)^{-1/n}B \in SU(n)$ and $\varphi_n((detB)^{1/n}, (detB)^{-1/n}B) = B$, i.e. φ_n is onto; (iii) let $(e^{-i\theta}, (a_{ij}))$ be in $ker(\varphi_n)$, i.e., $e^{-i\theta}(a_{ij}) = I$; then $a_{ij} = 0$ for $i \neq j$ and $a_{11} = \ldots = a_{nn} = e^{i\theta}$; since $det(a_{ij}) = e^{in\theta} = 1$, then $\theta = 2\pi m/n$ for $m = 0, 1, \ldots, n - 1$, and therefore $ker(\varphi_n) = \{(1, I), (e^{-i2\pi/n}, e^{i2\pi/n}I), \ldots, (e^{-i2\pi(n-1)/n}, e^{i2\pi(n-1)/n}I)\} \cong \mathbb{Z}_n$. Then one has an isomorphism of short exact sequences of Lie groups and Lie group homomorphisms given by the following diagram:

$$\begin{array}{ccccccccc}
1 & \to & \mathbb{Z}_n & \xrightarrow{\iota_n} & U(1) \times SU(n) & \xrightarrow{\varphi_n} & U(n) & \to & 1 \\
 & & \| & & \| & & \Phi_n \uparrow \cong & & \\
1 & \to & \mathbb{Z}_n & \xrightarrow{\iota_n} & U(1) \times SU(n) & \xrightarrow{p_n} & \dfrac{U(1) \times SU(n)}{\mathbb{Z}_n} & \to & 1
\end{array}$$

where ι_n is the inclusion $\iota_n(k) = (e^{-i2\pi k/n}, e^{i2\pi k/n}I)$, $k = 0, 1, \ldots, n - 1$, p_n is the canonical projection $p_n(z, A) = [z, A]$, and Φ_n is the *Lie group isomorphism* $\Phi_n([z, A]) = \varphi_n(z, A) = zA$.

On the other hand, d: $U(n) \to U(1)$ given by $A \mapsto d(A) := detA$ is a Lie group homomorphism with $ker(d) = SU(n)$. Then one has another isomorphism of short exact sequences of Lie groups and Lie group homomorphisms given by the following diagram:

$$\begin{array}{ccccccccc}
1 & \to & SU(n) & \xrightarrow{\iota} & U(n) & \xrightarrow{d} & U(1) & \to & 1 \\
 & & \| & & \| & & \cong \downarrow \psi & & \\
1 & \to & SU(n) & \xrightarrow{\iota} & U(n) & \xrightarrow{\pi} & \dfrac{U(n)}{SU(n)} & \to & 1
\end{array}$$

where ι is the inclusion $\iota(A) = A$, π is the canonical projection $\pi(B) = BSU(n)$, and ψ is the group isomorphism $\psi(z) = \left(\begin{smallmatrix} z & 0 \\ 0 & I \end{smallmatrix}\right) SU(n)$ where I is the $(n - 1) \times (n - 1)$ unit matrix. Since π is a principal $SU(n)-$bundle, then $U(n) \xrightarrow{d} U(1)$ is an $SU(n)$-principal bundle over S^1, and since the set of isomorphism classes of principal $SU(n)$-bundles over the 1-sphere $k_{SU(n)}(S^1)$ is in one-to-one correspondence with $\Pi_0(SU(n)) \cong 0$, then the bundle d is trivial. The global section σ: $U(1) \to U(n)$, $z \mapsto \sigma(z) := \left(\begin{smallmatrix} z & 0 \\ 0 & I \end{smallmatrix}\right)$ induces the $SU(n)$-*bundle isomorphism*

$$\begin{array}{ccc}
SU(n) & & SU(n) \\
\downarrow & & \downarrow \\
U(n) & \xrightarrow{\Psi_n} & U(1) \times SU(n) \\
{\scriptstyle d} \searrow & & \swarrow {\scriptstyle \pi_1} \\
 & U(1) &
\end{array}$$

given by $\Psi_n(B) = (z, A)$ with $z = detB$ and $A = \sigma(\bar{z})B$, where $\pi_1(z, A) = z$. Note that Ψ_n is *not* a Lie group homomorphism ($\Psi_n(BB') = (zz', \sigma(\bar{z})\sigma(\bar{z}')BB') \neq \Psi_n(B)\Psi_n(B')$ since for all $B \in U(n)$, $\sigma(\bar{z}')B = B\sigma(\bar{z}')$ only if $z' = 1$), but only a *diffeomorphism* of smooth manifolds; its inverse is given by $\Psi_n^{-1}(z, A) = \sigma(z)A$.

In summary, we have the commutative diagram

$$\begin{array}{ccc} & \dfrac{U(1) \times SU(n)}{\mathbb{Z}_n} & \\ \Phi_n \swarrow & & \searrow t_n \\ U(n) & \xrightarrow{\Psi_n} & U(1) \times SU(n) \end{array}$$

where $t_n = \Psi_n \circ \Phi_n$ is given by $t_n([z, A]) = (z^n, \sigma(\bar{z}^n)z A)$ and is a *diffeomorphism* of smooth manifolds, but *not* a Lie group isomorphism. So, φ_n: $U(1) \times SU(n) \to U(n)$ is an n-covering space of $U(n)$ by a space diffeomorphic to it. (This result is similar to that of the double covering of the circle, the $\mathbb{Z}_2$-bundle $S^1 \to S^1$, $z \mapsto z^2$.)

3. THE UNIVERSAL COVERING GROUP $\tilde{U}(N)$

Since topologically $U(n) \cong U(1) \times SU(n)$, then $\Pi_1(U(n)) \cong \Pi_1(U(1) \times SU(n)) \cong \Pi_1(U(1)) \oplus \Pi_1(SU(n)) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$, and so the universal covering group of $U(n)$, $\tilde{U}(n)$, is a principal $\mathbb{Z}$-bundle ξ: $\mathbb{Z} \to \tilde{U}(n) \to U(n)$. On the other hand, the universal principal $\mathbb{Z}$-bundle is $\xi_{\mathbb{Z}}$: $\mathbb{Z} \to \mathbb{R} \xrightarrow{exp} U(1)$ with $exp(t) = e^{i2\pi t}$. Then ξ is a pullback of $\xi_{\mathbb{Z}}$. We have found a natural map between $U(n)$ and $U(1)$, namely the determinant d. We shall prove that

$$\tilde{U}(n) \cong d^*(\mathbb{R}) \cong SU(n) \circledS \mathbb{R}$$

Remark 1. The first isomorphism was given by Fulton and Harris (1991), but not its relation to fiber bundle theory.

Proposition 1. Let $d^*(\xi_{\mathbb{Z}})$ be the pullback bundle of $\xi_{\mathbb{Z}}$ by the determinant d: $U(n) \to U(1)$:

$$\begin{array}{ccc} \mathbb{Z} & & \mathbb{Z} \\ \downarrow & & \downarrow \\ d^*(\mathbb{R}) & \xrightarrow{\pi_2} & \mathbb{R} \\ \pi_1\downarrow & & \downarrow exp \\ U(n) & \xrightarrow{d} & U(1) \end{array}$$

where π_1 and π_2 are the projections in the first and second factor, respectively. Then π_1 is the universal covering group of $U(n)$.

Proof. Since d and exp are group homomorphisms, $d^*(\mathbb{R})$ is a subgroup of $U(n) \times \mathbb{R}$, and clearly π_1 and π_2 are homomorphisms. The subgroup

$d^*(\mathbb{R})$ is the subset of $U(n) \times \mathbb{R}$ where the maps d and exp coincide, and $U(1)$ is Hausdorff, therefore $d^*(\mathbb{R})$ is closed. Since $U(n) \times \mathbb{R}$ is a Lie group, $d^*(\mathbb{R})$ is also a Lie group. Clearly, the maps π_1 and π_2 are smooth.

Now exp: $\mathbb{R} \to U(1)$ is a covering space with fiber $\mathbb{Z}$. Therefore π_1: $d^*(\mathbb{R}) \to U(n)$ is also a covering space with fiber $\mathbb{Z}$. Hence, we have a monomorphism $(\pi_1)_*$: $\Pi_1(d^*(\mathbb{R})) \to \Pi_1(U(n))$, and the quotient $\Pi_1(U(n))/(\pi_1)_*(\Pi_1(d^*(\mathbb{R})))$ is isomorphic to $\mathbb{Z}$. Since $\Pi_1(U(n)) \cong \mathbb{Z}$, then $\Pi_1(d^*(\mathbb{R})) \cong 0$. QED

Definition 1. Consider the Lie groups $SU(n)$ and $\mathbb{R}$. We define a smooth action $SU(n) \times \mathbb{R} \to SU(n)$ given by $(A, t) \mapsto A \cdot t := \sigma_{-t} A \sigma_t$, where $\sigma_t := \sigma(e^{i2\pi t})$. Using this action, we have the semidirect product $SU(n) \circledS \mathbb{R}$, whose underlying manifold is $SU(n) \times \mathbb{R}$, but with product given by the formula $(A, t)(A', t') := ((A \cdot t')A', t + t')$. One can easily check that this is a Lie group.

Theorem 1. The universal covering group of $U(n)$ is given by the map p: $SU(n) \circledS \mathbb{R} \to U(n)$, where $p(A, t) = \sigma(e^{i2\pi t})A$.

Proof. As mentioned above, $SU(n) \circledS \mathbb{R}$ is a Lie group and a simple calculation shows that p is a homomorphism. Since p is a composition of smooth maps, it is smooth.

Let ω: $SU(n) \circledS \mathbb{R} \to d^*(\mathbb{R})$ be the map given by $\omega(A, t) = (p(A, t), t)$. Clearly ω is smooth and one can easily verify that it is a homomorphism of Lie groups, whose inverse is given by $\omega^{-1}(B, t) = (\sigma(e^{-i2\pi t})B, t)$. Then we have the following commutative diagram:

$$\begin{array}{ccc} SU(n) \circledS \mathbb{R} & \xrightarrow{\omega} & d^*(\mathbb{R}) \\ {\scriptstyle p}\searrow & & \swarrow{\scriptstyle \pi_1} \\ & U(n) & \end{array}$$

By Proposition 1, $d^*(\mathbb{R})$ is the universal covering group of $U(n)$, therefore $SU(n) \circledS \mathbb{R}$ is also the universal covering group of $U(n)$. QED

Remark 2. In the literature one sometimes finds that the universal covering group of $U(n)$ is the direct sum of $SU(n)$ and $\mathbb{R}$ (Cornwell, 1984); however, by the theorem above, this is not the case.

Remark 3. For the particular case $n = 2$, Φ_2 in Section 2 says that as a group, $U(2)$ is constructed from the *unique* three spheres which are groups, namely S^0, S^1, and S^3, respectively, the unit real, complex, and quaternionic numbers (Aguilar *et al.*, 1998), and one has the commutative diagram

$$\begin{array}{ccc} & \dfrac{S^1 \times S^3}{S^0} & \\ {}^{\Phi_2}\swarrow & & \searrow^{t_2} \\ U(2) & \xrightarrow{\Psi_2} & S^1 \times S^3 \end{array}$$

where Φ_2 is a group isomorphism, and Ψ_2 and t_2 are diffeomorphisms. The explicit formulas for this case are the following: $p_2(z, A) \equiv [z, A] = \{(z, A), (-z, -A)\}$, $\Phi_2([z, A]) = zA$, and

$$\Psi_2\begin{pmatrix} a & b \\ -\bar{b}e^{i\lambda} & \bar{a}e^{i\lambda} \end{pmatrix} = \left(e^{i\lambda}, \begin{pmatrix} ae^{-i\lambda} & be^{-i\lambda} \\ -\bar{b}e^{i\lambda} & \bar{a}e^{i\lambda} \end{pmatrix}\right)$$

where $|a|^2 + |b|^2 = 1$ and $\lambda \in [0, 2\pi)$. [Another way to show the topological equivalence of $U(2)$ and $U(1) \times SU(2)$ is to start from the bundle $U(1) \to U(2) \to U(2)/U(1)$ and use the fact that $U(2)/U(1) \cong S^3$; since $\Pi_2(U(1)) \cong 0$, then the bundle is trivial, and so $U(2) \cong U(1) \times S^3$.]

The product in the universal covering $\tilde{U}(2) \cong SU(2) \circledS \mathbb{R}$ is given by

$$\left(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, t\right)\left(\begin{pmatrix} a' & b' \\ -\bar{b}' & \bar{a}' \end{pmatrix}, t'\right)$$
$$= \left(\begin{pmatrix} aa' - b\bar{b}'e^{-i2\pi t'} & ab' + b\bar{a}'e^{-i2\pi t'} \\ -\bar{a}\bar{b}' - \bar{b}a'e^{i2\pi t'} & \bar{a}\bar{a}' - \bar{b}b'e^{i2\pi t'} \end{pmatrix}, t + t'\right)$$

4. PROJECTIVE AND UNITARY REPRESENTATIONS

Let $\mathcal{H}$ be a complex Hilbert space with the standard norm topology $\|v\|^2 = \langle v, v\rangle$ for any $v \in \mathcal{H}$. Let $\hat{\mathcal{H}}$ be the projective space of its 1-dimensional subspaces with the quotient topology relative to the projection $v \mapsto \hat{v} := \mathbb{C}^* v$, for $v \neq 0$. Let $Aut(\hat{\mathcal{H}})$ be the group of automorphisms of $\hat{\mathcal{H}}$, where a bijection T: $\hat{\mathcal{H}} \to \hat{\mathcal{H}}$ is an automorphism if $\langle T(\hat{v}_1), T(\hat{v}_2)\rangle = \langle \hat{v}_1, \hat{v}_2\rangle$, where $\langle \hat{v}_1, \hat{v}_2\rangle$ (the transition probability in quantum mechanics) is given by $|\langle v_1, v_2\rangle|^2/\|v_1\|^2\|v_2\|^2$. Let $\tilde{\mathcal{U}}(\mathcal{H})$ be the group of unitary or antiunitary transformations of $\mathcal{H}$. Let π: $\tilde{\mathcal{U}}(\mathcal{H}) \to Aut(\hat{\mathcal{H}})$ be the projection $\pi(A)(\hat{v}) = \hat{A}(\hat{v}) := \hat{A}v$, and ι: $U(1) \to \tilde{\mathcal{U}}(\mathcal{H})$ the inclusion $\iota(z) = zId$. Then Wigner's theorem (Wigner, 1959) says that the following sequence is exact:

$$1 \to U(1) \xrightarrow{\iota} \tilde{\mathcal{U}}(\mathcal{H}) \xrightarrow{\pi} Aut(\hat{\mathcal{H}}) \to 1$$

i.e., any probability-preserving transformation of the projective Hilbert space is the image of a unitary or antiunitary transformation of the Hilbert space itself, and, moreover, if $\pi(A_1) = \pi(A_2)$, then $A_2 = e^{i\varphi}A_1$ with $\varphi \in [0, 2\pi)$ (Simms, 1968). If $\mathcal{U}(\mathcal{H})$ is the subgroup of $\tilde{\mathcal{U}}(\mathcal{H})$ of unitary operators, then the sequence

$$1 \to U(1) \overset{\iota}{\to} \mathcal{U}(\mathcal{H}) \overset{\pi}{\to} \mathcal{U}(\hat{\mathcal{H}}) \to 1$$

is also exact, where $\mathcal{U}(\hat{\mathcal{H}})$ is the image of $\mathcal{U}(\mathcal{H})$ under π, and is a subgroup of $Aut(\hat{\mathcal{H}})$. $\mathcal{U}(\mathcal{H})$ is given the *strong operator topology*, that is, the smallest topology which makes continuous the maps E_h: $\mathcal{U}(\mathcal{H}) \to \mathcal{H}$, $E_h(A) := A(h)$, for $h \in \mathcal{H}$, and $\mathcal{U}(\hat{\mathcal{H}})$ is given the quotient topology relative to the projection $A \mapsto \hat{A}$.

Definition 2. If X is a topological space, then f: $X \to \mathcal{U}(\mathcal{H})$ is said to be *strongly continuous* if it is continuous when $\mathcal{U}(\mathcal{H})$ is given the strong operator topology. Clearly, f is strongly continuous if and only if $E_h \circ f$ is continuous for each $h \in \mathcal{H}$.

On the other hand, Bargmann's theorem (Bargmann, 1954) says that if G is a connected and simply connected Lie group with second cohomology $H^2(Lie(G); \mathbb{R}) \cong 0$, then for any *projective representation* of G on $\mathcal{H}$, i.e., for any continuous group homomorphism ρ: $G \to \mathcal{U}(\hat{\mathcal{H}})$,there exists a *unitary representation* of G on $\mathcal{H}$, i.e., a strongly continuous group homomorphism $\hat{\rho} : G \to \mathcal{U}(\mathcal{H})$ such that $\pi \circ \hat{\rho} = \rho$ (Simms, 1968). $\hat{\rho}$ is called a *lifting* of ρ. Clearly, $\tilde{U}(n)$ satisfies the conditions of Bargmann's theorem, since by a theorem of Chevalley and Eilenberg (1948), the real cohomology of a compact and connected Lie group is isomorphic to the cohomology of its Lie algebra, and $H^2(U(n); \mathbb{R}) \cong 0$ (Itô, 1993). In Theorem 4 below, we give an explicit expression for the lifting $\hat{\rho}$. Moreover, in Proposition 3, we shall show that $\hat{\rho}$ is in fact a homomorphism of topological groups.

The following proposition gives a useful characterization of strongly continuous maps.

Proposition 2. Let G be a topological group and let f: $G \to \mathcal{U}(\mathcal{H})$ be a group homomorphism. Let $\bar{f}$: $G \times \mathcal{H} \to \mathcal{H}$ be the action given by $\bar{f}(g, h) := f(g)(h)$. Then $\bar{f}$ is continuous if and only if f is strongly continuous.

Proof. ($\Rightarrow$) Assume that $\bar{f}$ is continuous, and consider the composition $\bar{f} \circ \alpha_h$ with α_h: $G \to G \times \mathcal{H}$ given by $\alpha_h(g) = (g, h)$. Since α_h is clearly continuous, then $\bar{f} \circ \alpha_h$ is continuous. But $\bar{f} \circ \alpha_h = E_h \circ f$, i.e., f is strongly continuous.

($\Leftarrow$) Assume now that f is strongly continuous. Let $(g_0, h_0) \in G \times \mathcal{H}$. We shall prove that $\bar{f}$ is continuous at (g_0, h_0). For this, let $\{(g_\lambda, h_\lambda)\}_{\lambda \in \Lambda}$ be a net in $G \times \mathcal{H}$ which converges to (g_0, h_0). We will show that the net $\{\bar{f}(g_\lambda, h_\lambda)\}_{\lambda \in \Lambda}$ converges to $\bar{f}(g_0, h_0)$.

Let (g, h) be any point in $G \times \mathcal{H}$. Then we have the following inequality for the norm of $\bar{f}(g, h) - \bar{f}(g_0, h_0) = f(g)(h) - f(g_0)(h_0)$:

$$\begin{aligned}
&\|f(g)(h) - f(g_0)(h_0)\| \\
&\quad = \|f(g_0)((f(g_0))^{-1}f(g)(h) - h_0)\| \\
&\quad = \|f(g_0)(f(g_0^{-1}g)(h) - h_0)\| \\
&\quad = \|f(g_0^{-1}g)(h) - h_0\| \\
&\quad = \|f(g_0^{-1}g)(h - h_0) + f(g_0^{-1}g)(h_0) - h_0\| \\
&\quad \leq \|f(g_0^{-1}g)(h - h_0)\| + \|f(g_0^{-1}g)(h_0) - h_0\| \\
&\quad = \|h - h_0\| + \|f(g_0^{-1}g)(h_0) - h_0\| \qquad (*)
\end{aligned}$$

Notice that in the third and sixth steps we used the fact that f takes values in the unitary group. Now consider:

(i) Let μ_0 be the composition of the product in G and the inclusion $G \to G \times G$ given by $g \mapsto (g_0^{-1}, g)$. Since μ_0 is continuous and $\mu_0(g_0) = e$, and the net $\{g_\lambda\}_{\lambda \in \Lambda} \to g_0$, then the net $\{\mu_0(g_\lambda) = g_0^{-1}g_\lambda\}_{\lambda \in \Lambda} \to e$.

(ii) Since f is strongly continuous, then the composition $G \xrightarrow{f} \mathcal{U}(\mathcal{H}) \xrightarrow{E_{h_0}} \mathcal{H}$ is continuous; then $E_{h_0} \circ f \circ \alpha$ given by $g \mapsto f(g_0^{-1}g)(h_0)$ is also continuous. Since $\{g_\lambda\}_{\lambda \in \Lambda} \to g_0$, then $\{f(g_0^{-1}g_\lambda)(h_0)\}_{\lambda \in \Lambda} \to h_0$.

(iii) Let $\varepsilon > 0$; then there exists $\lambda_0 \in \Lambda$ such that for all $\lambda > \lambda_0$, $\|f(g_0^{-1}g_\lambda)(h_0) - h_0\| < \varepsilon/2$; on the other hand, since $\{h_\lambda\}_{\lambda \in \Lambda} \to h_0$, then there exists $\lambda_1 \in \Lambda$ such that for all $\lambda > \lambda_1$, $\|h_\lambda - h_0\| < \varepsilon/2$. Since Λ is a directed set, then there exists $\bar{\lambda} \in \Lambda$ such that $\bar{\lambda} \geq \lambda_0$ and $\bar{\lambda} \geq \lambda_1$. Then for all $\lambda \geq \bar{\lambda}$ we have, by the inequality $(*)$, $\|f(g_\lambda)(h_\lambda) - f(g_0)(h_0)\| \leq \|h_\lambda - h_0\| + \|f(g_0^{-1}g_\lambda)(h_0) - h_0\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore, $\{\bar{f}(g_\lambda, h_\lambda)\}_{\lambda \in \Lambda}$ converges to $f(g_0, h_0)$, i.e., $\bar{f}$ is continuous. QED

Remark 4. We have two canonical subgroups of $SU(n)$ Ⓢ $\mathbb{R}$. The subgroup $\{(A, 0) | A \in SU(n)\}$ which is normal, and the subgroup $\{(I, t) | t \in \mathbb{R}\}$, which is not normal; the intersection of both subgroups is trivial, so that any element (A, t) can be written uniquely as the product $(A, t) = (I, t)(A, 0)$. In the theory of unitary representations of semidirect products (Sternberg, 1994), one asumes that the normal subgroup is Abelian. This is the case of the proper orthochronous Poincaré group which is the semidirect product of $\mathbb{R}^4$ and $SO^0(3, 1)$, where $\mathbb{R}^4$ is normal and Abelian. However, in our case, $SU(n)$ is not Abelian.

Proposition 3. The group $\mathcal{U}(\mathcal{H})$, with the strong topology, is a topological group.

Proof. (i) Let μ: $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ be the product in $\mathcal{U}(\mathcal{H})$, i.e., $\mu(R, S) = R \circ S$. We shall show that, for each $h \in \mathcal{H}$, the composition $E_h \circ \mu$ is continuous. For this, let $\{(R_\lambda, S_\lambda)\}_{\lambda \in \Lambda}$ be a net in $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$ which converges (in the strong topology) to (R_0, S_0). Let $\varepsilon > 0$; since $\{R_\lambda\}_{\lambda \in \Lambda} \to R_0$ and $\{S_\lambda\}_{\lambda \in \Lambda} \to S_0$, there exist λ_1 and λ_2 in Λ such that $\|(R_\lambda - R_0)(S_0(h))\| < \varepsilon/2$, whenever $\lambda \geq \lambda_1$, and $\|(S_\lambda - S_0)(h)\| < \varepsilon/2$ whenever $\lambda \geq \lambda_2$. Now,

$$\begin{aligned} &\|R_\lambda \circ S_\lambda(h) - R_0 \circ S_0(h)\| \\ &\quad = \|(R_\lambda - R_0)(S_0(h)) + R_\lambda(S_\lambda(h) - S_0(h))\| \\ &\quad \leq \|(R_\lambda - R_0)(S_0(h))\| + \|R_\lambda(S_\lambda(h) - S_0(h))\| \\ &\quad = \|(R_\lambda - R_0)(S_0(h))\| + \|(S_\lambda - S_0(h))\| \end{aligned}$$

Let $\bar{\lambda}$ be an element in Λ such that $\bar{\lambda} \geq \lambda_1$ and $\bar{\lambda} \geq \lambda_2$. If $\lambda \geq \bar{\lambda}$, then $\|R_\lambda \circ S_\lambda(h) - R_0 \circ S_0(h)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $\{R_\lambda \circ S_\lambda(h)\} \to R_0 \circ S_0(h)$ and then $E_h \circ \mu$ is continuous.

(ii) It is known that the weak and the strong topologies for the set of bounded operators on $\mathcal{H}$ coincide in $\mathcal{U}(\mathcal{H})$. To show that the map $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ given by $R \mapsto R^{-1}$ is continuous, we shall use the weak topology. Let $\{R_\lambda\}_{\lambda \in \Lambda}$ be a net in $\mathcal{U}(\mathcal{H})$ which converges, in the weak topology, to R_0. Let h and h' be elements in $\mathcal{H}$. Then, given $\varepsilon > 0$, there exists λ_0 in Λ such that $|\langle h, (R_\lambda - R_0)(h')\rangle| < \varepsilon$ whenever $\lambda > \lambda_0$. Now, since the operators are unitary, $R^* = R^{-1}$ and $R_0^* = R_0^{-1}$. Hence $(R_\lambda - R_0)^* = R_\lambda^* - R_0^* = R_\lambda^{-1} - R_0^{-1}$, and the inequality above can be written as $\varepsilon > |\langle h, (R_\lambda - R_0)(h')\rangle| = |\langle (R_\lambda - R_0)^*(h), h'\rangle| = |\langle (R_\lambda^{-1} - R_0^{-1})(h), h'\rangle|$ whenever $\lambda > \lambda_0$. Therefore $\{R_\lambda^{-1}\}_{\lambda \in \Lambda} \to R_0^{-1}$, so that the map $R \mapsto R^{-1}$ is continuous at any point R_0. QED

Notice that since π: $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\hat{\mathcal{H}})$ is an open map, then $\pi \times \pi$ is also open. Therefore $\mathcal{U}(\hat{\mathcal{H}})$ is also a topological group.

Remark 5. Naimark (1964) showed that the composition of bounded operators on an infinite dimensional Hilbert space is not strongly continuous. However, when the composition is restricted to the unitary operators, the above proposition shows that the composition is indeed continuous, contrary to what is claimed in Simms (1968, p. 10).

Corollary 1. The action $\mathcal{U}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ given by $(A, h) \mapsto A(h)$ is continuous.

Proof. In Proposition 2, take $G = \mathcal{U}(\mathcal{H})$ and $f = id$. QED

Note. Since we always use the strong topology on $\mathcal{U}(\mathcal{H})$, a strongly continuous map with codomain $\mathcal{U}(\mathcal{H})$ will be called continuous for short.

5. CLASSIFICATION OF PROJECTIVE REPRESENTATIONS

In this section we give a bijection between the set of projective representations of $U(n)$ and the set of equivalence classes of certain unitary representations of $SU(n) \circledS \mathbb{R}$.

Definition 3. Let λ_1, λ_2: $SU(n) \circledS \mathbb{R} \to U(1)$ be continuous homomorphisms. We define $\lambda_1\lambda_2$: $SU(n) \circledS \mathbb{R} \to U(1)$ by $(\lambda_1\lambda_2)(a) = \lambda_1(a)\lambda_2(a)$. This map is continuous since it is the composition of the following continuous maps:

$$SU(n) \circledS \mathbb{R} \xrightarrow{\Delta} (SU(n) \circledS \mathbb{R}) \times (SU(n) \circledS \mathbb{R}) \xrightarrow{\lambda_1 \times \lambda_2} U(1) \times U(1) \xrightarrow{\nu} U(1)$$

where $\Delta(a) = (a, a)$ and ν is the product in $U(1)$. Moreover, since $U(1)$ is Abelian, the product ν is a homomorphism. Therefore $\lambda_1\lambda_2$ is also a homomorphism because it is a composite of homomorphisms. The rule $(\lambda_1, \lambda_2) \mapsto \lambda_1\lambda_2$ gives a group structure to the set of continuous homomorphisms, with identity $1(a) = 1$ and inverses $\lambda^{-1}(a) = \lambda(a)^{-1}$. We denote this group by $(SU(n) \circledS \mathbb{R})^* \equiv \mathbf{U_0}$. It is the *group of one-dimensional (irreducible) representations of* $SU(n) \circledS \mathbb{R}$. (In Proposition 9, we will show that $\mathbf{U_0}$ is isomorphic to $\mathbb{R}$.)

Let ι: $\mathbb{Z} \to SU(n) \circledS \mathbb{R}$ be the inclusion $\iota(k) = (I, k)$; clearly $\iota(\mathbb{Z}) = ker(p)$.

Definition 4. We denote by $\mathcal{U}(SU(n) \circledS \mathbb{R}) \equiv \mathbf{U}$ the set of unitary representations β: $SU(n) \circledS \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that $\beta(\iota(\mathbb{Z})) \subset U(1)Id$, and by $hom(U(n), \mathcal{U}(\hat{\mathcal{H}})) \equiv \mathbf{P}$ the set of projective representations of $U(n)$. We define a map Q: $\mathbf{U} \to \mathbf{P}$ as follows: we associate to β the homomorphism $Q(\beta)$: $U(n) \to \mathcal{U}(\hat{\mathcal{H}})$ given by $Q(\beta)(p(a)) = \pi \circ \beta(a)$, i.e., $Q(\beta)$ is the homomorphism in the quotient groups, induced by β.

Lemma 1. The map Q: $\mathbf{U} \to \mathbf{P}$ is well defined and it is surjective.

Proof. The homomorphism β maps $\iota(\mathbb{Z}) = ker(p)$ into $U(1)Id = ker(\pi)$; therefore the homomorphism $Q(\beta)$ is well defined. Since p is a covering space, it is an open map, so that $U(n)$ has the quotient topology. Hence $Q(\beta)$ is continuous and it is then a projective representation.

Now let ρ: $U(n) \to \mathcal{U}(\hat{\mathcal{H}})$ be a projective representation. As we did before, we can use Bargmann's theorem for the representation $\rho \circ p$ to get a representation $\hat{\rho}$: $SU(n) \circledS \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that $\pi \circ \hat{\rho} = \rho \circ p$; clearly $\hat{\rho}(\iota(\mathbb{Z})) \subset U(1)Id$. Hence $\hat{\rho} \in \mathbf{U}$. Since $\pi \circ \hat{\rho} = Q(\hat{\rho}) \circ p$, then $\rho \circ p = Q(\hat{\rho}) \circ p$, and, since p is surjective, $\rho = Q(\hat{\rho})$. Therefore Q is surjective. QED

In order to study the map Q, we will define an action of the group $\mathbf{U_0}$ on $\mathbf{U}$.

Proposition 4. There is a free action $\mathbf{U_0} \times \mathbf{U} \to \mathbf{U}$ given by $(\lambda, \beta) \mapsto \lambda \cdot \beta$, where $(\lambda \cdot \beta)(a) = \lambda(a)Id \circ \beta(a)$.

Proof. (i) The map $\lambda \cdot \beta$ can be written as the following composite:

$$SU(n) \circledS \mathbb{R} \xrightarrow{\Delta} (SU(n) \circledS \mathbb{R}) \times (SU(n) \circledS \mathbb{R})$$
$$\xrightarrow{\lambda\times\beta} U(1) \times \mathcal{U}(\mathcal{H}) \xrightarrow{\iota\times Id} \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \xrightarrow{\mu} \mathcal{U}(\mathcal{H})$$

By Proposition 3, μ is continuous, therefore $\lambda \cdot \beta$ is the composition of continuous maps, so it is continuous.

(ii) Since $(\lambda \cdot \beta)(a_1a_2) = \lambda(a_1a_2)Id \circ \beta(a_1a_2) = \lambda(a_1)\lambda(a_2)Id \circ \beta(a_1) \circ \beta(a_2) = \lambda(a_1)Id \circ \beta(a_1) \circ \lambda(a_2)Id \circ \beta(a_2) = (\lambda \cdot \beta)(a_1) \circ (\lambda \cdot \beta)(a_2)$, then $\lambda \cdot \beta$ is a homomorphism.

(iii) Now recall that the elements of the form $\lambda(a)Id$ belong to $ker(\pi)$ and that $\beta(ker(p)) \subset ker(\pi)$. Therefore, if $a \in ker(p)$, then $\pi \circ (\lambda \cdot \beta)(a) = \pi(\lambda(a)Id \circ \beta(a)) = \pi(\lambda(a)Id) \circ \pi(\beta(a)) = \widehat{Id}$. Hence $(\lambda \cdot \beta)(ker(p)) \subset ker(\pi)$. These facts show that $\lambda \cdot \beta$ is an element of $\mathbf{U}$.

(iv) An easy calculation, similar to (ii), shows that $(\lambda, \beta) \mapsto \lambda \cdot \beta$ is an action.

(v) Assume that $\lambda \cdot \beta = \beta$; then for all a in $SU(n) \circledS \mathbb{R}$, we have that $\lambda(a)Id \circ \beta(a) = \beta(a)$; therefore $\lambda(a)Id = Id$, and $\lambda(a) = 1$, i.e., $\lambda = 1$. Hence the action is free. QED

Theorem 2. Let β_1 and β_2 be elements in $\mathbf{U}$. Then $Q(\beta_1) = Q(\beta_2)$ if and only if there exists an element λ in $\mathbf{U_0}$ such that $\beta_1 = \lambda \cdot \beta_2$. In other words, the fibers of Q are precisely the orbits of the action of the group $\mathbf{U_0}$.

Proof. ($\Rightarrow$) Assume that $Q(\beta_1) = Q(\beta_2)$. Let $\tilde{\lambda}$ be the following composite:

$$SU(n) \circledS \mathbb{R} \xrightarrow{\Delta} (SU(n) \circledS \mathbb{R}) \times (SU(n) \circledS \mathbb{R})$$
$$\xrightarrow{\beta_1\times\beta_2} \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \xrightarrow{\tilde{\mu}} \mathcal{U}(\mathcal{H})$$

where Δ is the diagonal map and $\tilde{\mu}(A, B) = A \circ B^{-1}$. By Proposition 3, $\tilde{\mu}$ is continuous, hence $\tilde{\lambda}$ is continuous. Since $\pi(\beta_1(a) \circ \beta_2(a)^{-1}) = \pi(\beta_1(a)) \circ \pi(\beta_2(a))^{-1} = Q(\beta_1)(p(a)) \circ Q(\beta_2)(p(a))^{-1} = \hat{I}d$, then $\tilde{\lambda}(a) = \beta_1(a) \circ \beta_2(a)^{-1} \in ker(\pi) = U(1)Id$. Using this fact, we have that $\tilde{\lambda}(a_1a_2) = \beta_1(a_1a_2) \circ \beta_2(a_1a_2)^{-1} = \beta_1(a_1) \circ \beta_1(a_2) \circ \beta_2(a_2)^{-1} \circ \beta_2(a_1)^{-1} = \beta_1(a_1) \circ \beta_2(a_1)^{-1} \circ \beta_1(a_2) \circ \beta_2(a_2)^{-1} = \tilde{\lambda}(a_1) \circ \tilde{\lambda}(a_2)$, i.e., $\tilde{\lambda}$ is a continuous homomorphism such that $\tilde{\lambda}(SU(n) \circledS \mathbb{R}) \subset U(1)Id$ and $\beta_1(a) = \tilde{\lambda}(a) \circ \beta_2(a)$. Since $\iota\colon U(1) \to \mathcal{U}(\mathcal{H})$ given by $\iota(z) = zId$ is a topological embedding and a homomorphism, we have a continuous homomorphism λ: $SU(n) \circledS \mathbb{R} \to U(1)$ such that $\iota \circ \lambda = \tilde{\lambda}$, and clearly $\beta_1 = \lambda \cdot \beta_2$.

($\Leftarrow$) Assume that $\beta_1 = \lambda \cdot \beta_2$. Since $\lambda(a)Id \subset ker(\pi)$, then we have that $Q(\beta_1)(p(a)) = Q(\lambda \cdot \beta_2)(p(a)) = \pi((\lambda \cdot \beta_2)(a)) = \pi(\lambda(a)Id \circ \beta_2(a)) = \pi(\beta_2(a)) = Q(\beta_2)(p(a))$. Therefore $Q(\beta_1) = Q(\beta_2)$. QED

Corollary 2. Let ρ: $U(n) \to \mathcal{U}(\hat{\mathcal{H}})$ be a projective representation. Then there is a bijection between the group $\mathbf{U_0}$ and $Q^{-1}(\{\rho\})$.

Proof. By the theorem, $Q^{-1}(\{\rho\})$ coincides with the orbit of any element β in $Q^{-1}(\{\rho\})$. But by Proposition 4, the action is free, so that the orbit of any point is in bijective correspondence with the group $\mathbf{U_0}$. QED

Remark 6. Once we choose an element β in $Q^{-1}(\{\rho\})$, we have a bijection from $\mathbf{U_0}$ to $Q^{-1}(\{\rho\})$ given by $\lambda \mapsto \lambda \cdot \beta$. So, in general there is no canonical way to identify $Q^{-1}(\{\rho\})$ with the group $\mathbf{U_0}$. However, when ρ is the trivial representation $\hat{c}$ given by $\hat{c}(A) = \hat{Id}$, there is a canonical element c in $Q^{-1}(\{\hat{c}\})$, namely, the trivial unitary representation given by $c(a) = Id$, and in this case $Q^{-1}(\{\hat{c}\})$ can be identified with the representations $\mathbf{U_0}$ through $\lambda \mapsto \lambda \cdot c$.

Remark 7. The bijection between the set $\mathbf{P}$ of projective representations of $U(n)$ and the orbits of $\mathbf{U}$ under the action of $\mathbf{U_0}$ is given by the map $[\beta] \mapsto Q(\beta)$.

Finally, the following theorem, together with Theorem 2 above, allows a classification of the projective representations of $U(n)$ in terms of unitary representations of $SU(n)$ and one-parameter unitary groups, satisfying certain conditions.

Theorem 3. There is a canonical bijection between $\mathbf{U}$ and the set of pairs (f_1, f_2) which satisfy the following conditions:

(i) f_1: $SU(n) \to \mathcal{U}(\mathcal{H})$ and f_2: $\mathbb{R} \to \mathcal{U}(\mathcal{H})$ are both continuous homomorphisms.

(ii) $f_2(1) \in U(1)Id$.

(iii) $f_1(A \cdot t) = f_2(t)^{-1} \circ f_1(A) \circ f_2(t)$ for all A in $SU(n)$ and t in $\mathbb{R}$.

Proof. Let β: $SU(n) \circledS \mathbb{R} \to \mathcal{U}(\mathcal{H})$ be an element in $\mathbf{U}$. Let ι_1: $SU(n) \to SU(n) \circledS \mathbb{R}$ and ι_2: $\mathbb{R} \to SU(n) \circledS \mathbb{R}$ be the canonical inclusions. Then we associate to β the pair $(\beta \circ \iota_1, \beta \circ \iota_2)$; clearly both are continuous homomorphisms. Since $\beta(\iota(\mathbb{Z})) \subset U(1)Id$, then $\beta \circ \iota_2(1) = \beta \circ \iota(1)$ is an element of $U(1)Id$. Since β is a homomorphism, and any element (A, t) in $SU(n) \circledS \mathbb{R}$ can be written as $(A, t) = (I, t)(A, 0) = \iota_2(t)\iota_1(A)$, then $\beta((A, t)(A', t')) = \beta \circ \iota_2(t) \circ \beta \circ \iota_1(A) \circ \beta \circ \iota_2(t') \circ \beta \circ \iota_1(A')$; but $\beta((A, t)(A', t')) = \beta((A \cdot t')A', t + t') = \beta(\iota_2(t + t')\iota_1((A \cdot t')A')) = \beta \circ \iota_2(t) \circ \beta \circ \iota_2(t') \circ \beta \circ \iota_1(A \cdot t') \circ \beta \circ \iota_1(A')$, then $\beta \circ \iota_1(A \cdot t') = \beta \circ \iota_2(t')^{-1} \circ \beta \circ \iota_1(A) \circ \beta \circ \iota_2(t')$.

Conversely, given a pair (f_1, f_2) satisfying (i)–(iii), we define $f: SU(n) \circledS \mathbb{R} \to \mathcal{U}(\mathcal{H})$ by $f(A, t) = f_2(t) \circ f_1(A)$. This map can be written as the composite

$$SU(n) \circledS \mathbb{R} \xrightarrow{f_1 \times f_2} \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \xrightarrow{s} \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \xrightarrow{\mu} \mathcal{U}(\mathcal{H})$$

where $s(A, B) = (B, A)$. By Proposition 3, μ is continuous, hence f is continuous. Now, using property (iii), we can write $f((A, t)(A', t')) = f((A \cdot t')A', t + t') = f_2(t) \circ f_2(t') \circ f_1(A \cdot t') \circ f_1(A') = f_2(t) \circ f_2(t') \circ f_2(t')^{-1} \circ f_1(A) \circ f_2(t') \circ f_1(A') = f_2(t) \circ f_1(A) \circ f_2(t') \circ f_1(A') = f(A, t) \circ f(A', t')$, i.e., f is a homomorphism; and $f(I, n) = f_2(n) \circ f_1(I) = f_2(n) \circ Id = f_2(n)$, which is an element of $U(1)Id$, by property (ii).

Finally, one construction is the inverse of the other. Indeed, let $f_1 = \beta \circ \iota_1$ and $f_2 = \beta \circ \iota_2$; then $f(A, t) = \beta \circ \iota_2(t) \circ \beta \circ \iota_1(A) = \beta(\iota_2(t)) \circ \beta(\iota_1(A)) = \beta(I, t) \circ \beta(A, 0) = \beta((I, t)(A, 0)) = \beta(A, t)$, i.e., $f = \beta$; conversely, given f constructed from the pair (f_1, f_2), define the pair $(f \circ \iota_1, f \circ \iota_2)$; then $f \circ \iota_1(A) = f(A, 0) = f_2(0) \circ f_1(A) = Id \circ f_1(A) = f_1(A)$ and $f \circ \iota_2(t) = f(I, t) = f_2(t) \circ f_1(I) = f_2(t) \circ Id = f_2(t)$, i.e., $f \circ \iota_1 = f_1$ and $f \circ \iota_2 = f_2$. QED

6. UNITARY REPRESENTATIONS ASSOCIATED TO A PROJECTIVE REPRESENTATION

In this section we shall construct all the unitary representations of $\tilde{U}(n)$ associated to a projective representation of $U(n)$.

Proposition 5. Let G be a connected simple Lie group and let K be a Lie group such that $dim\ G > dim\ K$. Let $\gamma: G \to K$ be a continuous homomorphism. Then γ is trivial, i.e., $\gamma(g) = e$, for all $g \in G$.

Proof. By Warner (1983), any continuous homomorphism between Lie groups is smooth, so we can assume that γ is smooth. Consider the differential of γ at the identity, $d\gamma: \mathcal{G} \to \mathcal{K}$, where $\mathcal{G}$ and $\mathcal{K}$ are, respectively, the Lie algebras of G and K. Since G is simple, $ker(d\gamma)$ is either 0 or $\mathcal{G}$. Suppose that $ker(d\gamma) = 0$; then $dim\ \mathcal{G} = dim\ d\gamma(\mathcal{G}) \le dim\ \mathcal{K}$, which is a contradiction because $dim\ \mathcal{G} > dim\ \mathcal{K}$. Therefore $ker(d\gamma) = \mathcal{G}$, i.e., $d\gamma \equiv 0$, and since G is connected, by Warner (1983), γ is trivial. QED

Remark 8. Let $\mathbb{R}^* = hom(\mathbb{R}, U(1))$ be the group of continuous homomorphisms $\chi: \mathbb{R} \to U(1)$, where $(\chi_1\chi_2)(t) = \chi_1(t)\chi_2(t)$. Then there is an isomorphism from $\mathbb{R}$ to $\mathbb{R}^*$ given by $r \mapsto \chi_r$, where $\chi_r(t) = e^{2\pi irt}$. This can be shown as follows. Recall that if G is a Lie group, then we have a function from its Lie algebra $\mathcal{G}$ to the set of one-parameter subgroups of G, given by $X \mapsto exp(tX)$. Let $\psi: \mathbb{R} \to G$ be a one-parameter subgroup. Since ψ is the maximal integral curve of the left-invariant vector field $X = \dot{\psi}(0)$ starting from e, then $\psi(t) = exp(tX)$. Therefore the function $X \mapsto exp(tX)$ is a bijection. Taking $G = U(1)$ and using the fact that continuous homomorphisms between Lie groups are smooth (Warner, 1983), we get a bijection between $i\mathbb{R}$ and $\mathbb{R}^*$, which in this case is clearly an isomorphism of groups. If we also consider the isomorphism from $\mathbb{R}$ to $i\mathbb{R}$ given by $r \mapsto 2\pi ir$, then we get an isomorphism $\mathbb{R} \cong \mathbb{R}^*$ mapping r to χ_r.

Definition 5. Let G be a topological group. We denote by $hom(G, \mathcal{U}(\mathcal{H}))$ the set of continuous homomorphisms from G to $\mathcal{U}(\mathcal{H})$, i.e., the set of unitary representations of G, and by $hom(G, \mathcal{U}(\hat{\mathcal{H}}))$ the set of continuous homomorphisms from G to $\mathcal{U}(\hat{\mathcal{H}})$, i.e., the set of projective representations of G.

Proposition 6. Let G be a connected, simply connected, simple Lie group such that $H^2(\mathcal{G}, \mathbb{R}) = 0$. Then the map $\pi: \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\hat{\mathcal{H}})$ induces a bijection $\pi_*: hom(G, \mathcal{U}(\mathcal{H})) \to hom(G, \mathcal{U}(\hat{\mathcal{H}}))$, where $\pi_*(\beta) = \pi \circ \beta$.

Proof. By Bargmann's theorem, given $\rho \in hom(G, \mathcal{U}(\hat{\mathcal{H}}))$, there is an element $\tilde{\rho}$ in $hom(G, \mathcal{U}(\mathcal{H}))$ such that $\pi \circ \tilde{\rho} = \rho$, i.e., $\pi_*(\tilde{\rho}) = \rho$. Therefore π_* is surjective. To show that π_* is injective, consider β_1 and β_2 in $hom(G, \mathcal{U}(\mathcal{H}))$ and assume that $\pi_*(\beta_1) = \pi_*(\beta_2)$. Then for all elements g in G we have that $\pi(\beta_1(g)) = \pi(\beta_2(g))$. Hence $\pi(\beta_1(g)\beta_2(g)^{-1}) = \hat{Id}$. Define $\gamma: G \to \mathcal{U}(\mathcal{H})$ by $\gamma(g) = \beta_1(g)\beta_2(g)^{-1}$. By Proposition 3, $\mathcal{U}(\mathcal{H})$ is a topological group, so that γ is continuous. Furthermore, $\gamma(G) \subset U(1)Id \cong U(1)$, whose elements commute with any element in $\mathcal{U}(\mathcal{H})$. Then $\gamma(g_1g_2) = \beta_1(g_1g_2)\beta_2(g_1g_2)^{-1} = \beta_1(g_1)\beta_1(g_2)\beta_2(g_2)^{-1}\beta_2(g_1)^{-1} = \beta_1(g_1)\beta_2(g_1)^{-1}\beta_1(g_2)\beta_2(g_2)^{-1} = \gamma(g_1)\gamma(g_2)$. Therefore $\gamma: G \to U(1)$ is a continuous homomorphism, so by Proposition 5, γ is trivial. Then $1 = \gamma(g) = \beta_1(g)\beta_2(g)^{-1}$ for all g in G; thus $\beta_1 = \beta_2$ and π_* is injective. QED

Definition 6. We define an action $\cdot: \mathbb{R} \times hom(\mathbb{R}, \mathcal{U}(\mathcal{H})) \to hom(\mathbb{R}, \mathcal{U}(\mathcal{H}))$ by $(r \cdot \lambda)(t) = e^{2\pi irt}Id \circ \lambda(t)$.

Lemma 2. The action is well defined and free.

Proof. (i) $(r \cdot \lambda)(t)$ is in $\mathcal{U}(\mathcal{H})$ since it is the composition of elements of $\mathcal{U}(\mathcal{H})$.

(ii) $r \cdot \lambda: \mathbb{R} \to \mathcal{U}(\mathcal{H})$ is continuous since it is the composite:

$$\mathbb{R} \to U(1) \times \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$$

$$t \mapsto (e^{2\pi irt}, \lambda(t)) \mapsto (e^{2\pi irt}Id, \lambda(t)) \mapsto e^{2\pi irt}Id \circ \lambda(t)$$

(iii) $r \cdot \lambda$ is a homomorphism because $(r \cdot \lambda)(t_1 + t_2) = e^{2\pi ir(t_1+t_2)}Id \circ \lambda(t_1 + t_2) = e^{2\pi irt_1}Id \circ \lambda(t_1) \circ e^{2\pi irt_2}Id \circ \lambda(t_2) = (r \cdot \lambda)(t_1) \circ (r \cdot \lambda)(t_2)$.

(iv) It is an action because $(0 \cdot \lambda)(t) = \lambda(t)$ and $(r_1 + r_2) \cdot \lambda(t) = e^{2\pi i(r_1+r_2)t}Id \circ \lambda(t) = (e^{2\pi i r_1 t}e^{2\pi i r_2 t})Id \circ \lambda(t) = e^{2\pi i r_1 t}Id \circ e^{2\pi i r_2 t}Id \circ \lambda(t) = r_1 \cdot (r_2 \cdot \lambda)(t)$.

(v) Assume that $r \cdot \lambda = \lambda$; then $(r \cdot \lambda)(t) = e^{2\pi irt}Id \circ \lambda(t) = \lambda(t)$ for all t in $\mathbb{R}$.

This implies that $e^{2\pi irt} = 1$ for all t in $\mathbb{R}$, therefore $r = 0$. QED

Proposition 7. Let λ_1, λ_2: $\mathbb{R} \to \mathcal{U}(\mathcal{H})$ be continuous homomorphisms. Then $\pi \circ \lambda_1 = \pi \circ \lambda_2$ if and only if there exists r in $\mathbb{R}$ such that $\lambda_1 = r \cdot \lambda_2$.

Proof. Assume that $r \cdot \lambda_1 = \lambda_2$. Then $\lambda_2(t) = e^{2\pi irt}Id \circ \lambda_1(t)$ and $\pi(\lambda_2(t)) = \pi(e^{2\pi irt}Id) \circ \pi(\lambda_1(t)) = \pi(\lambda_1(t))$. Therefore, $\pi \circ \lambda_1 = \pi \circ \lambda_2$.

Assume that $\pi \circ \lambda_1 = \pi \circ \lambda_2$. Then $\pi(\lambda_1(t)\lambda_2(t)^{-1}) = \hat{Id}$ and we have that $\lambda_1(t)\lambda_2(t)^{-1}$ is in $ker(\pi) = U(1)Id$, for all t in $\mathbb{R}$. Define ψ: $\mathbb{R} \to U(1)$ by $\psi(t) = \lambda_1(t)\lambda_2(t)^{-1}$. By Proposition 3, $\mathcal{U}(\mathcal{H})$ is a topological group, so ψ is continuous. Furthermore, $\psi(t_1 + t_2) = \lambda_1(t_1 + t_2)\lambda_2(t_1 + t_2)^{-1} = \lambda_1(t_1) \circ \lambda_1(t_2) \circ \lambda_2(t_2)^{-1} \circ \lambda_2(t_1)^{-1} = \lambda_1(t_1) \circ \lambda_2(t_1)^{-1} \circ \lambda_1(t_2) \circ \lambda_2(t_2)^{-1} = \psi(t_1) \circ \psi(t_2)$. Therefore, ψ is a continuous homomorphism. By Remark 8, there exists a unique real number r such that $\psi(t) = e^{2\pi irt}$. Hence $\lambda_1(t) = \psi(t)\lambda_2(t) = e^{2\pi irt}\, Id \circ \lambda_2(t) = (r \cdot \lambda_2)(t)$, so that $\lambda_1 = r \cdot \lambda_2$. QED

Proposition 8. The map π: $\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\hat{\mathcal{H}})$ induces a surjection

$$\pi_*: \quad hom(\mathbb{R}, \mathcal{U}(\mathcal{H})) \to hom(\mathbb{R}, \mathcal{U}(\hat{\mathcal{H}}))$$

given by $\pi_*(\lambda) = \pi \circ \lambda$, whose fibers are in one-to-one correspondence with $\mathbb{R}$.

Proof. Since $\mathbb{R}$ satisfies the hypothesis of Bargmann's theorem, given any one-parameter subgroup δ: $\mathbb{R} \to \mathcal{U}(\hat{\mathcal{H}})$, there exists $\tilde{\delta}$ in $hom(\mathbb{R}, \mathcal{U}(\mathcal{H}))$ such that $\pi \circ \tilde{\delta} = \delta$, i.e., $\pi_*(\tilde{\delta}) = \delta$, so π_* is surjective.

By Proposition 7, given δ in $hom(R, \mathcal{U}(\hat{\mathcal{H}}))$, $\pi_*^{-1}(\{\delta\})$ coincides with an orbit of the action of $\mathbb{R}$ on $hom(\mathbb{R}, \mathcal{U}(\mathcal{H}))$. By Lemma 2, this action is free, so each orbit is in one-to-one correspondence with the group $\mathbb{R}$. QED

Remark 9. Let λ be in $hom(\mathbb{R}, \mathcal{U}(\mathcal{H}))$. Since λ is a strongly continuous one-parameter unitary group, then, by Stone's theorem (Reed and Simon, 1972), there exists a unique Hermitian (though not necessarily bounded) operator H on $\mathcal{H}$ such that $\lambda(t) = e^{iHt}$.

Proposition 9. $\mathbf{U_0} = \{f$: $SU(n) \circledS \mathbb{R} \to U(1) | f$ is a continuous homomorphism$\}$ is isomorphic to $\mathbb{R}$.

Proof. We will give first an isomorphism between $\mathbf{U_0}$ and $\mathbb{R}^* = \{\chi$: $\mathbb{R} \to U(1) | \chi$ is a continuous homomorphism$\}$. Let f be in $\mathbf{U_0}$ and consider $f \circ \iota_1$: $SU(n) \to U(1)$. Then by Proposition 5 we know that $f \circ \iota_1$ is constant. So we define a function F: $\mathbf{U_0} \to \mathbb{R}^*$ by $F(f) = f \circ \iota_2$: $\mathbb{R} \to U(1)$. Given χ in

$\mathbb{R}^*$, consider the map $\overline{\chi}$: $SU(n) \circledS \mathbb{R} \to U(1)$ given by $\overline{\chi}(A, t) = \chi(t)$. Since $\overline{\chi} = \chi \circ \pi_2$, where π_2: $SU(n) \circledS \mathbb{R} \to \mathbb{R}$ is the projection, and both χ and π_2 are continuous homomorphisms, then $\overline{\chi}$ is in $\mathbf{U_0}$. Furthermore, $F(\overline{\chi}) = \overline{\chi} \circ \iota_2 = \chi$, so F is surjective.

Now let f_1, f_2 be in $\mathbf{U_0}$; then $F(f_1 f_2)(t) = (f_1 f_2) \circ \iota_2(t) = f_1(\iota_2(t))\, f_2(\iota_2(t)) = f_1 \circ \iota_2(t) f_2 \circ \iota_2(t) = F(f_1)(t) F(f_2)(t)$. Hence F is a homomorphism.

Let f be in $\mathbf{U_0}$ and assume that $F(f)$ is trivial, i.e., $F(f)(t) = 1$ for all t in $\mathbb{R}$. Any element (A, t) in $SU(n) \circledS \mathbb{R}$ can be written as $(A, t) = (I, t)(A, 0) = \iota_2(t)\iota_1(A)$; therefore $f(A, t) = f \circ \iota_2(t) f \circ \iota_1(A)$. Since $F(f) = f \circ \iota_2$ is trivial, then $f(A, t) = f \circ \iota_1(A)$, but we saw that $f \circ \iota_1$ is also trivial, so f is trivial, and then F is injective. Therefore F is an isomorphism.

Finally, by Remark 8, there is an isomorphism from $\mathbb{R}$ to $\mathbb{R}^*$ given by $r \mapsto \chi_r$, where $\chi_r(t) = e^{2\pi irt}$. Using both isomorphisms, we get an isomorphism from $\mathbb{R}$ to $\mathbf{U_0}$ given by $r \mapsto f_r$: $SU(n) \circledS \mathbb{R} \to U(1)$, where $f_r(A, t) = e^{2\pi irt}$. QED

Theorem 4. Let ρ: $U(n) \to \mathcal{U}(\hat{\mathcal{H}})$ be a projective representation. Then there is a bijection between $Q^{-1}(\{\rho\})$ and the set $\{(\mathring{\hat{\rho}} \circ \iota_1, r \cdot (\mathring{\hat{\rho}} \circ \iota_2) | r \in \mathbb{R}\}$, where $\mathring{\hat{\rho}}$: $SU(n) \circledS \mathbb{R} \to \mathcal{U}(\mathcal{H})$ is any fixed representation in $Q^{-1}(\{\rho\})$ and where $r \cdot (\mathring{\hat{\rho}} \circ \iota_2)(t) = e^{2\pi irt} Id \circ \mathring{\hat{\rho}}(Id, t)$. The bijection is given by $(\mathring{\hat{\rho}} \circ \iota_1, r \cdot (\mathring{\hat{\rho}} \circ \iota_2)) \mapsto \hat{\rho}$: $SU(n) \circledS \mathbb{R} \to \mathcal{U}(\mathcal{H})$, where $\hat{\rho}(A, t) = e^{2\pi irt} Id \circ \mathring{\hat{\rho}}(A, t)$.

Proof. We first recall that the existence of $\mathring{\hat{\rho}}$ is given by Bargmann's theorem. Now, by Theorem 3, there is a canonical bijection between $Q^{-1}(\{\rho\})$ and the set $\mathcal{A}$ of pairs (f_1, f_2) which satisfy the conditions (i)–(iii) plus the following condition (iv) to ensure that the elements belong to $Q^{-1}(\{\rho\})$:

(iv) $Q(f) = \rho$, where $f(A, t) = f_2(t) \circ f_1(A)$.

Let $\mathcal{B}$ be the set of pairs $\{(\mathring{\hat{\rho}} \circ \iota_1, r \cdot (\mathring{\hat{\rho}} \circ \iota_2) | r \in \mathbb{R}\}$ in the statement of the theorem. We will show that $\mathcal{A} = \mathcal{B}$.

Let $(\mathring{\hat{\rho}} \circ \iota_1, r \cdot (\mathring{\hat{\rho}} \circ \iota_2))$ be in $\mathcal{B}$. Clearly $\mathring{\hat{\rho}} \circ \iota_1$ is a continuous homomorphism, and the same is true for $r \cdot (\mathring{\hat{\rho}} \circ \iota_2)$ because $\mathcal{U}(\mathcal{H})$ is a topological group, so (i) is satisfied.

Since $\mathring{\hat{\rho}}$ is an element of $\mathbf{U}$, then $\mathring{\hat{\rho}} \circ \iota_2(1) = \mathring{\hat{\rho}}(Id, 1)$ belongs to $U(1)Id$, hence $r \cdot (\mathring{\hat{\rho}} \circ \iota_2)(1) = e^{2\pi ir} Id \circ \mathring{\hat{\rho}}(Id, 1)$ is in $U(1)Id$, so (ii) is satisfied.

By Theorem 3, $\mathring{\hat{\rho}} \circ \iota_1(A \cdot t) = \mathring{\hat{\rho}} \circ \iota_2(t)^{-1} \circ \mathring{\hat{\rho}} \circ \iota_1(A) \circ \mathring{\hat{\rho}} \circ \iota_2(t)$. Since the elements of the form $e^{2\pi irt} Id$ commute with any operator, the right-hand side is equal to $\mathring{\hat{\rho}} \circ \iota_2(t)^{-1} \circ e^{-2\pi irt} Id \circ \mathring{\hat{\rho}} \circ \iota_1(A) \circ e^{2\pi irt} Id \circ \mathring{\hat{\rho}} \circ \iota_2(t) = r \cdot (\mathring{\hat{\rho}} \circ \iota_2)(t)^{-1} \circ \mathring{\hat{\rho}} \circ \iota_1(A) \circ r \cdot (\mathring{\hat{\rho}} \circ \iota_2)(t)$. Therefore the pair in $\mathcal{B}$ satisfies (iii).

Condition (iv) is also fulfilled since $Q(\hat{\rho})(p(A, t)) = \pi \circ \hat{\rho}(A, t) = \pi(e^{2\pi irt} Id) \circ \pi(\mathring{\hat{\rho}}(Id, t)) \circ \pi(\mathring{\hat{\rho}}(A, 0)) = \rho(p(Id, t)) \circ \rho(p(A, 0)) = \rho(p((Id, t)(A, 0))) = \rho(p(A, t))$. Therefore $\mathcal{B} \subset \mathcal{A}$.

Conversely, let (f_1, f_2) be an element in $\mathcal{A}$. Consider $\rho \circ \iota: SU(n) \to \mathcal{U}(\hat{\mathcal{H}})$; since $\pi \circ \mathring{\hat{\rho}} \circ \iota_1 = \rho \circ p \circ \iota_1 = \rho \circ \iota$, then $\mathring{\hat{\rho}} \circ \iota_1$ is a lifting of $\rho \circ \iota$. Let A be in $SU(n)$; then by (iv) we have that $Q(f)(\iota(A)) = Q(f)(p(A, 0)) = \pi(f(A, 0)) = \pi(f_2(0) \circ f_1(A)) = \pi(f_1(A)) = \rho \circ \iota(A)$, i.e., f_1 is also a lifting of $\rho \circ \iota$. By Proposition 6, $f_1 = \mathring{\hat{\rho}} \circ \iota$. Now consider $\rho \circ \alpha: \mathbb{R} \to \mathcal{U}(\hat{\mathcal{H}})$, since $\pi \circ \mathring{\hat{\rho}} \circ \iota_2 = \rho \circ p \circ \iota_2 = \rho \circ \alpha$, then $\mathring{\hat{\rho}} \circ \iota_2$ is a lifting of $\rho \circ \alpha$. By (iv) we have that $Q(f)(\alpha(t)) = Q(f)(p(Id, t)) = \pi(f(Id, t)) = \pi(f_2(t) \circ f_1(Id)) = \pi(f_2(t)) = \rho(\alpha(t))$, i.e., f_2 is also a lifting of $\rho \circ \alpha$. By Proposition 7, there exists r in $\mathbb{R}$ such that $f_2 = r \cdot (\mathring{\hat{\rho}} \circ \iota_2)$. Therefore $(f_1, f_2) = (\mathring{\hat{\rho}} \circ \iota, r \cdot (\mathring{\hat{\rho}} \circ \iota_2))$ and then $\mathcal{A} \subset \mathcal{B}$.

Finally, by Theorem 3, the bijection between $\mathcal{A} = \mathcal{B}$ and $Q^{-1}(\{\rho\})$ is given by mapping a pair (f_1, f_2) to f, where $f(A, t) = f_2(t) \circ f_1(A)$; therefore $(\mathring{\hat{\rho}} \circ \iota_1, r \cdot (\mathring{\hat{\rho}} \circ \iota_2))$ is mapped to $\hat{\rho}$, where $\hat{\rho}(A, t) = e^{2\pi irt}\, Id \circ \mathring{\hat{\rho}}(A, t)$. QED

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